

We present English translation of the classical article of Hermann Amadeus Schwarz (1843–1921) *”Proof of the theorem that a surface area of a ball is smaller than of any other body of the same volume”* which was published in 1884, in Proceedings of the Königlische Gesellschaft der Wissenschaften and the Georg-Augusts-Universität, Göttingen.

We preserved the author notations throughout the text and tried to follow his grammar construction of the sentences. One editorial comment in the footnote at the page 4 is related to the (possible) misprint in the original German text. The other three comments in the footnotes at the pages 6, 7, 8 are given to warn about ambiguities in definition of different notions and entities.

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**Proof of the theorem that a surface area of a ball
is smaller than
of any other body of the same volume
by H. A. Schwartz**

Proceedings of the Königlische Gesellschaft der Wissenschaften
and the Georg-Augusts-Universität, Göttingen, 1884, pages 1-13

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In order to prove the theorem that *a surface area of a ball is smaller than that of any other body of the same volume*, several different approaches were used, which are mainly based on the condition that *among all bodies of same volume, there exists one that has a minimal surface area*. As long as the condition, is not proved none of the aforementioned methods are valid to be used in order to prove the main theorem.

Trying to prove the above mentioned *theorem* for bodies which surface is formed by a finite number of finite analytical surface pieces, I was led to a method which does not seem to be exposed to the objection of missing rigor. The proof presented in this manuscript, is based on the repetitive use of an method that has been used by Mr. *Weierstrass* in his lectures on calculus of variation. I'm indebted to one of his oral presentations for the knowledge about this method.

§1

Let \mathcal{U} be a non-spherical body which surface \mathfrak{B} is formed by a finite number of pieces of analytical surfaces assumed to be free of singularities.

The points of the surface \mathfrak{B} shall be related to a right-angular coordinate system, which is chosen such that no part of \mathfrak{B} is parallel to the yz -plane of the coordinate system. Let x_0 be the smallest and x_1 be the largest of all accessible values of the coordinate x .

Let an arbitrary point P that belongs to \mathfrak{B} and does not belong to any of its *edges* has a right-angular coordinates x, y, z . Construct a normal to \mathfrak{B} at P and fix its positive direction such that at point P its direction inclines from the outside to the inside w.r.t. the body \mathcal{U} . The angle between the positive direction of the normal and the positive direction of the x -axis shall be denoted by ξ .

Through this point P construct a plane \mathfrak{C}_x which is parallel to the yz -plane. This plane has generally

one or more curves \mathfrak{C} in common with the surface \mathfrak{B} . A totality \mathfrak{C}_x of these curves shall be considered as one curve, while dy, dz stand for coordinates of an element of \mathfrak{C}_x coming from point P with length ds . One can determine that the initial points for measuring the arc length s on the curves \mathfrak{C}_x ($x_0 < x < x_1$) form one or more analytical curves that belong to the surface \mathfrak{B} .

The values of x and s shall be chosen as independent variables to define the coordinates x, y, z of any point on the surface \mathfrak{B} .

The abovementioned assumptions imply that it is always possible to cut the surface \mathfrak{B} by a finite number of planes parallel to the yz -plane into a finite number of either bowl-shaped or ring-shaped partial surfaces, such that for each of them the coordinates y and z of any point are unique and generally continuous functions of two variables x and s . If the curve \mathfrak{C}_x is made of several separate closed curves, then one has to assign a specific order as well as starting points for the measurement of the arc length. The value of s shall be defined in such a way that if $U_1(x), U_2(x), \dots, U_n(x)$ is the *length* of the first, second, \dots , n -th closed parts of the curve \mathfrak{C}_x , then s takes all the values from 0 up $U_1(x)$ for the first part, $U_1(x)$ to $U_1(x) + U_2(x)$ for the second, \dots , $U_1(x) + U_2(x) + \dots + U_{n-1}(x)$ to $U(x)$ for the last one, where $U(x)$ denotes the total length of all parts of \mathfrak{C}_x .

Therefore the following equations hold:

$$\frac{\partial x}{\partial s} = 0, \quad \left(\frac{\partial y}{\partial s} \right)^2 + \left(\frac{\partial z}{\partial s} \right)^2 = 1.$$

Assume that a growth of s along each segment of the curve \mathfrak{C}_x was chosen such that

$$A = \frac{\partial y}{\partial x} \frac{\partial z}{\partial s} - \frac{\partial z}{\partial x} \frac{\partial y}{\partial s}, \quad B = -\frac{\partial z}{\partial s}, \quad C = \frac{\partial y}{\partial s},$$

where A, B, C are coordinates of a line segment which direction is the same as the direction of the *normal* to the surface \mathfrak{B} for every point P that does not belong to any *edge* of the surface \mathfrak{B} .

Under this condition, the integral along the curve \mathfrak{C}_x

$$\int_0^{U(x)} \frac{1}{2} \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) ds = Q(x)$$

is the area of the surface that consists of one or more pieces, which include all points that are inside of body \mathfrak{U} and the plane \mathfrak{C}_x and none else.

The volume V of the body \mathfrak{U} is given by following equation

$$\int_{x_0}^{x_1} Q(x) dx = V.$$

The size of an element dS on the surface \mathfrak{B} and the size dT of an orthogonal projection of dS onto the yz -plane of the coordinate system are given by equations ¹

$$dS = \sqrt{1 + A^2} \, dx ds = \frac{1}{\sin \xi} dx ds, \quad dT = \cos \xi \, dS = A dx ds = \cot \xi \, dx ds.$$

From the geometric meaning of $Q(x)$ and the integral along the curve \mathfrak{C}_x

$$\int_{s=0}^{s=U(x)} \cot \xi \, dx ds = dx \int_0^{U(x)} \cot \xi \, ds = dx \int_0^{U(x)} A ds$$

we obtain equation

$$dQ(x) = Q'(x) dx = dx \int_0^{U(x)} A ds.$$

The area S of the surface \mathfrak{B} and the volume V of the body \mathfrak{U} therefore read

$$S = \int_{x_0}^{x_1} dx \int_0^{U(x)} \sqrt{1 + \left(\frac{\partial y}{\partial x} \frac{\partial z}{\partial s} - \frac{\partial z}{\partial x} \frac{\partial y}{\partial s} \right)^2} ds = \int_{x_0}^{x_1} dx \int_0^{U(x)} \sqrt{1 + A^2} ds,$$

$$V = \int_{x_0}^{x_1} dx \int_0^{U(x)} \frac{1}{2} \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) ds = \int_{x_0}^{x_1} Q(x) dx.$$

§2

There exists a possibility that for one or more values of x the entity

$$\frac{\partial y}{\partial x} \frac{\partial z}{\partial s} - \frac{\partial z}{\partial x} \frac{\partial y}{\partial s} = A$$

reaches a value $Q'(x)/U(x)$ which is independent of s .

Under this condition there exists x such that

$$\int_0^{U(x)} \sqrt{1 + A^2} \, ds = \sqrt{U^2(x) + Q'^2(x)}.$$

But in any other case it holds that

$$\int_0^{U(x)} \sqrt{1 + A^2} \, ds > \sqrt{U^2(x) + Q'^2(x)}.$$

In order to prove this statement, set

$$\int_0^s A ds = t, \quad \int_0^s \sqrt{1 + A^2} ds = \int_0^s \sqrt{ds^2 + dt^2}$$

and determine an angle ω by equation

$$\cos \omega = \frac{s ds + t dt}{\sqrt{s^2 + t^2} \sqrt{ds^2 + dt^2}}$$

¹In the original German text instead of $\sqrt{1 + A^2}$ there is erroneously written $\sqrt{1 + A}$.

resulting in formula

$$d \left(\int_0^s \sqrt{ds^2 + dt^2} - \sqrt{s^2 + t^2} \right) = (1 - \cos \omega) \sqrt{ds^2 + dt^2},$$

which geometric interpretation is self-explanatory and therefore does not require any further remarks.

Since $(1 - \cos \omega)$ is non-negative and can vanish everywhere in the interval $0 < s < U(x)$, only when A is independent of s , integrating both sides of the equation (given above) between $s = 0$ and $s = U(x)$ one obtains

$$(I.a) \quad \int_0^{U(x)} \sqrt{1 + A^2} ds \geq \sqrt{U^2(x) + Q'^2(x)}.$$

Here the *equality* can be reached only when the values of x , A , and, therefore, ξ are independent of s .

(I.a) implies

$$(I.b) \quad S \geq \int_{x_0}^{x_1} \sqrt{U^2(x) + Q'^2(x)} dx.$$

The *equality* can be reached only when in the entire interval $x_0 < x < x_1$ it holds that $A = \cot \xi$ is a function of x alone.

§3

It may occur that for one or more values of x the closed curve \mathfrak{C}_x is formed by a single *circular arc* of radius r . Under this condition we obtain for values of x :

$$r^2 \pi = Q(x), \quad U^2(x) = (2r\pi)^2 = 4Q(x)\pi, \quad \sqrt{U^2(x) + Q'^2(x)} = \sqrt{4Q(x)\pi + Q'^2(x)}.$$

In any other case though it holds that

$$U^2(x) \geq 4Q(x)\pi.$$

To prove this assertion, in other words, to show that *the perimeter of a circular surface is smaller than that of any other planar figure of the same area*, we can proceed as follows.

In order to handle the case when the curve \mathfrak{C}_x consists of several closed curves similarly as in the case when \mathfrak{C}_x is given by a single closed curve, one can assume that the integral taken along a given *closed curve*, that lies inside the plane \mathfrak{C}_x ,

$$\int \frac{1}{2} (ydz - zdy)$$

does not change its value, when the curve is translated in the plane without changing its shape. Since $ds = \sqrt{dy^2 + dz^2}$ does not change during this operation, it can be used to translate all closed curves produced by intersection of the plane \mathfrak{C}_x with surface \mathfrak{B} , which produce \mathfrak{C}_x . We translate these curves

in such a manner that the point on each curve, where s has its largest or smallest value, is the same as the point O_x of intersection of the plane \mathfrak{E}_x with the x -axis.

By moving the pieces of the curve \mathfrak{C}_x and combining them into a single polygon chain – such that along this chain s grows monotonically, taking all values in the interval $0 < s < U(x)$ – one produces a single closed curve, which shall be called $\bar{\mathfrak{C}}_x$. The length of this curve is $U(x)$.

If we denote by y and z the second and the third coordinates of any point P on $\bar{\mathfrak{C}}_x$ that is determined by the value of s , then these variables are unique and *continuous* functions of s in the interval $0 < s < U(x)$ and equal to zero at the ends of this interval.

The integral along the curve $\bar{\mathfrak{C}}_x$

$$\int_0^{U(x)} \frac{1}{2} \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) ds$$

has a value of $Q(x)$.

We shall now introduce notations

$$\rho = \sqrt{y^2 + z^2}, \quad \mathfrak{F} = \int_0^s \frac{1}{2} \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) ds,$$

and consider a *circular segment* with the chord belonging to the plane \mathfrak{E}_x along the line segment $O_x P$, which is constructed in such manner that along the segment arc in the direction from O_x to P the integral

$$\int_0^s (y dz - z dy)$$

has the value \mathfrak{F} .

Denote,

r , positive or negative (according to the sign of \mathfrak{F}) value of the radius of the circular segment²;

ϕ , positive or negative value of a half of the central angle of the circular segment measured in radians;

\mathfrak{L} , length of the arc. The following equations hold:

$$\mathfrak{F} = r^2(\phi - \sin \phi \cos \phi), \quad \rho = 2r \sin \phi, \quad \mathfrak{L} = 2r\phi.$$

To simplify the analysis, we can assume, without loss of generality, that both ρ and \mathfrak{F} will never equal to 0 *simultaneously* for values of s *inside* the interval $0 < s < U(x)$. In the case when both ρ and \mathfrak{F} equal to 0 at s_0 in the interval $0 < s < U(x)$, but are not 0 at the same time for any value of s in the interval $s_0 < s < U(x)$, we can limit the analysis of the interval $0 < s < U(x)$ to the interval $s_0 < s < U(x)$. Then it holds that

$$U(x) > \sqrt{4Q(x)\pi}$$

²In beginning of §3 the variable r is defined by equality $r^2\pi = Q(x)$, as a radius of circular arc which forms the closed curve \mathfrak{C}_x .

a fortiori.

To determine the value of ϕ in the interval $-\pi \leq \phi \leq \pi$, we obtain a transcendent equation

$$f(\phi) = \frac{\phi - \sin \phi \cos \phi}{2 \sin^2 \phi} = \frac{2\mathfrak{F}}{\rho^2}.$$

Since the derivative

$$f'(\phi) = \frac{\tan \phi - \phi}{\tan \phi \sin^2 \phi}$$

is neither negative nor 0 for all values of ϕ in the above interval, while the function $f(\phi)$ takes all values from $-\infty$ to $+\infty$ in that interval, then for every value of s in the given interval the equation

$$f(\phi) = \frac{2\mathfrak{F}}{\rho^2}$$

has only one root which value continuously changes with the value of s .

The variable r that has the same sign as ϕ and \mathfrak{F} is thus defined completely. The value of $1/r$ changes continuously with s .

The curve $\bar{\mathfrak{C}}_x$ intersects with the line segment $O_x P$ as well as with the arc of the previously constructed circular segment at point P . If we denote by ω an angle³ formed by the tangents to the curve $\bar{\mathfrak{C}}_x$ and to the circular arc that connects O_x and P at the point P then following equations hold:

$$\begin{aligned} \cos \omega &= \frac{1}{\rho} \left[\left(y \frac{\partial y}{\partial s} + z \frac{\partial z}{\partial s} \right) \cos \phi + \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) \sin \phi \right], \\ \sin \omega &= \frac{1}{\rho} \left[\left(y \frac{\partial y}{\partial s} + z \frac{\partial z}{\partial s} \right) \sin \phi - \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) \cos \phi \right], \\ d\mathfrak{F} &= \frac{1}{2} \left(y \frac{\partial z}{\partial s} - z \frac{\partial y}{\partial s} \right) ds, \quad d\rho = \frac{1}{\rho} \left(y \frac{\partial y}{\partial s} + z \frac{\partial z}{\partial s} \right) ds, \\ d\mathfrak{L} &= \cos \omega ds, \quad \frac{\tan \phi - \phi}{\tan \phi \sin^2 \phi} d(1/r) = -\frac{2 \sin \omega}{\rho^2} ds. \end{aligned}$$

We therefore obtain an equation:

$$d(s - \mathfrak{L}) = (1 - \cos \omega) ds.$$

Since s takes all values from 0 to $U(x)$ while continuously growing and the term $1 - \cos \omega$ is nonnegative and only under a certain condition it *everywhere* equals 0, an equation holds

$$s - \mathfrak{L} = \int_0^s (1 - \cos \omega) ds,$$

from which we deduce that the *length* $s = U(x)$ of the curve $\bar{\mathfrak{C}}_x$, is generally *larger* than \mathfrak{L} that corresponds to the value $s = U(x)$.

³In §2 ω is defined by equation $\cos \omega = (sds + tdt)/(\sqrt{s^2 + t^2}\sqrt{ds^2 + dt^2})$.

For $s = U(x)$, ρ turns to 0, \mathfrak{F} reaches $Q(x)$, ϕ becomes π because $Q(x)$ is positive and the circular segment is replaced by a circular surface with radius $\sqrt{Q(x)/\pi}$, and \mathfrak{L} equals $\sqrt{4Q(x)\pi}$.

From the previous analysis we find following relationship:

$$(II.a) \quad U(x) \geq \sqrt{4Q(x)\pi}.$$

It is important to note that, in accordance with the statement made above, the *equality* takes place if and only if in the interval $0 < s < U(x)$ the quantity r has a value of $\sqrt{Q(x)/\pi}$, independent of s . It means that the curve \mathfrak{C}_x which belongs to the plane \mathfrak{E}_x is a *circular arc* with radius $\sqrt{Q(x)/\pi}$. Only under this condition the value of the terms $1 - \cos \omega$ and $\sin \omega$ is 0 in the whole interval $0 < s < U(x)$.

Combining formula (I.b) with (II.a) we find:

$$(II.b) \quad S \geq \int_{x_0}^{x_1} \sqrt{4Q(x)\pi + Q'^2(x)} dx.$$

It should be noted that the *equality* is reached only when the body \mathfrak{U} is a *body of rotation* with rotation axis parallel to the x -axis of the coordinate system.

§4

Consider a *body of rotation* \mathfrak{D} , which rotation axis coincides with the x -axis of the coordinate system, with a radius r of the parallel circle that lies in the plane \mathfrak{E}_x given by equations

$$r^2\pi = Q(x), \quad r \geq 0, \quad x_0 \leq x \leq x_1.$$

Then it appears that

$$2r\pi dr = Q'(x)dx, \quad 2r\pi\sqrt{dx^2 + dr^2} = \sqrt{4Q(x)\pi + Q'^2(x)} dx.$$

The surface area of the body of rotation \mathfrak{D} is given by integral

$$\int_{x_0}^{x_1} \sqrt{4Q(x)\pi + Q'^2(x)} dx.$$

Introduce the notation ⁴

$$\mathfrak{B} = \int_{x_0}^{x_1} Q(x) dx$$

and consider a *spherical segment* that lies on the negative side of the plane \mathfrak{E}_x , which volume is equal to \mathfrak{B} . The plane boundary has an area $r^2\pi = Q(x)$ and coincides with the parallel circle of \mathfrak{D} . The area of the curved boundary of the spherical segment, the so called *spherical cap*, shall be called \mathfrak{H} , its radius is R , and one quarter of the central angle in radians is ψ . The angle that the tangential plane to

⁴In §1 \mathfrak{B} denotes a surface of a non-spherical body \mathfrak{U} .

the spherical surface makes with the tangential plane to the body of rotation in the point of the common parallel circle will be denoted by ω and the length dl of the meridian segment of the body of rotation is equal to $\sqrt{dx^2 + dr^2}$.

With these assumptions, following equations hold

$$\begin{aligned}\mathfrak{B} &= \frac{4}{3}R^3\pi\sin^4\psi(\sin^2\psi + 3\cos^2\psi), \\ r &= 2R\sin\psi\cos\psi, \\ \mathfrak{H} &= 4R^2\pi\sin^2\psi, \\ d\mathfrak{B} &= r^2\pi dx, \quad d\mathfrak{H} = 2r\pi\cos\omega dl, \\ \cos\omega &= \cos 2\psi\frac{dr}{dl} + \sin 2\psi\frac{dx}{dl}, \\ \sin\omega &= \cos 2\psi\frac{dx}{dl} - \sin 2\psi\frac{dr}{dl}. \\ (1 + \tan^2\omega)^2 d\left(\frac{1}{R}\right) &= \frac{4\sin\psi}{r^2}dl,\end{aligned}$$

The value of ψ can be determined from the equation

$$f(\psi) = \tan\psi + \frac{1}{3}\tan^3\psi = \frac{2B}{r^3\pi},$$

under the condition that

$$0 \leq \psi \leq \pi/2.$$

Since the derivative of the function $f(\psi)$

$$f'(\psi) = (1 + \tan^2\psi)^2$$

is always positive, the function $f(\psi)$ takes all values from 0 to ∞ while growing continuously, whereas ψ takes all values from 0 to $\pi/2$ growing continuously as well. For this reason the equation

$$f(\psi) = \frac{2\mathfrak{B}}{r^3\pi}$$

in the given interval has one and only one root, which value continuously changes with x .

Since the value of ψ is unique, we can say the same about R due to the equation $r = 2R\sin\psi\cos\psi$.

From the equation

$$d\left[\int_{x_0}^x 2r\pi\sqrt{1 + (dr/dx)^2}dx - \mathfrak{H}\right] = 2r\pi(1 - \cos\omega)dl,$$

we obtain

$$\int_{x_0}^x \sqrt{4Q(x)\pi + Q'^2(x)}dx \geq \mathfrak{H}.$$

If now we set $x = x_1$, then r turns to 0, ψ reaches $\pi/2$, \mathfrak{B} becomes V , R results in the value of $\sqrt[3]{3V/4\pi}$ and \mathfrak{H} becomes $\sqrt[3]{36V^2\pi}$.

Furthermore we obtain

$$(III.a) \quad \int_{x_0}^{x_1} \sqrt{4Q(x)\pi + Q'^2(x)} dx \geq \sqrt[3]{36V^2\pi}.$$

The *equality* is reached only if in the interval $x_0 < x < x_1$, $\cos \omega = 1$, $\sin \omega = 0$ and, therefore, R has a value independent of x . In this case though it follows from the above equations that the body of rotation \mathfrak{D} is a *ball*.

If that happens, then the body \mathfrak{U} cannot be a body of rotation which axis of rotation is parallel to the x -axis of the coordinate system; \mathfrak{U} would have to be a ball as well (this possibility was already eliminated in the beginning).

Combining formulas (II.b) and (III.a), it appears that

$$S \geq \int_{x_0}^{x_1} \sqrt{4Q(x)\pi + Q'^2(x)} dx \geq \sqrt[3]{36V^2\pi}.$$

Under the given condition, *equality* can be reached on one but never on both sides of the above inequality at the same time. It is clear that

$$(III.b) \quad S \geq \sqrt[3]{36V^2\pi}.$$

By this it seems to me that the following theorem is proven rigorously:

The ball has a smaller surface than any other body of same volume which surface is formed by a finite number of finite pieces characterized as an algebraic surface in every point.

§5

The previous examination includes any finite body \mathfrak{U} which surface is formed by a finite number of analytical surfaces.

The condition "the surface of \mathfrak{U} is formed from a finite number of *analytical* surfaces" is sufficient but, as one can easily be convinced, not necessary for the conclusion that "the analyzed body \mathfrak{U} has, if it is not a ball itself, a greater surface than a ball of same volume". The theorem mentioned in the beginning of this manuscript holds at all times even without this limiting condition, if the surface of \mathfrak{U} is formed by a finite number of finite pieces each one having at every point a *unique tangential plane* which changes continuously with the point location.

To prove this, we just have to show that for every body \mathfrak{U} that was constructed this way and is not a ball, there exists a *polyhedron* \mathfrak{U}^* bounded by a finite number of planar surfaces which has the *same* volume but a *smaller* surface area than \mathfrak{U} .

The conclusion that the body \mathfrak{U} has a greater surface area than a ball with the same volume is proven by applying the previously performed analysis (§§1-4) on the body \mathfrak{U}^* a fortiori.

Now, there exists an abstract in Steiner's paper "Ueber Maximum und Minimum bei den Figuren in der Ebene, auf der Kugelfläche und im Raume überhaupt" [Gesammelte Werke Band II. Seite 300-306] that provides the means to construct for every non-spherical body \mathfrak{U} with the mentioned properties, the surface S and the volume V , another body \mathfrak{U}' which volume V' is *equal* to V whereas the surface area S' is *smaller* than S .

If we consider such a body \mathfrak{U}' and introduce two non-zero variables ϵ and η , not subjected to any restrictions due to their smallness, there is an infinite number of ways to construct a *polyhedron* \mathfrak{U}'' bounded by planar surfaces. We construct \mathfrak{U}'' in such a way that if we denote its surface area by S'' and the volume by V'' then the differences $S'' - S'$ and $V'' - V'$ in absolute values are smaller than $\epsilon S'$ and $\eta V'$, respectively. If we now construct a polyhedron \mathfrak{U}'' that is *similar* to \mathfrak{U}^* , which volume V^* is *equal* to the volume of body \mathfrak{U} , then the surface area S^* of this polyhedron is smaller than

$$(1 + \epsilon) \sqrt[3]{(1 - \eta)^{-2}} S'.$$

If we choose ϵ and η from the beginning such that

$$\frac{(1 + \eta)^3}{(1 - \eta)^2} < \left(\frac{S}{S'} \right)^3,$$

then we find that $S^* < S$.

We conclude that it is possible, as asserted, for any non-spherical body \mathfrak{U} with the mentioned properties, to construct a polyhedron \mathfrak{U}^* bounded by a finite number of plane surfaces, which has the *same* volume and a *smaller* surface area than \mathfrak{U} .

We conclude that out of all bodies of the same volume with surfaces of the mentioned properties, the ball has the smallest surface area.